

MATHEMATICS

LOCAL AND GLOBAL HYPERCENTRALITY  
AND SUPERSOLUBILITY I

BY

REINHOLD BAER

*Dedicated to Hans Freudenthal on the occasion of his 60th birthday*

(Communicated by Prof. H. D. KLOOSTERMAN at the meeting of September 25, 1965)

If  $e$  is a group theoretical property, then a group is termed locally  $e$ , if all its finitely generated subgroups are  $e$ -groups. It is easy to see that locally  $e$ -subgroups are always contained in maximal locally  $e$ -subgroups; and this rather fundamental property is in general not shared by  $e$  itself. Thus there arise the following two general problems.

- I. To give criteria for a locally  $e$ -group to be an  $e$ -group.
- II. To investigate the class of groups whose locally  $e$ -subgroups are  $e$ -groups.

These two general problems we are going to discuss in two special and closely related instances: the hypercentral and the supersoluble groups.

Here we term the group  $G$   $\left\{ \begin{smallmatrix} \text{hypercentral} \\ \text{supersoluble} \end{smallmatrix} \right\}$ , if every epimorphic image, not 1, of  $G$  possesses a  $\left\{ \begin{smallmatrix} \text{center element} \\ \text{cyclic normal subgroup} \end{smallmatrix} \right\}$ , not 1. It is clear that every hypercentral group is supersoluble and it is known that the commutator subgroup of a supersoluble group is hypercentral.

Assume now that every primary abelian subgroup and every torsionfree abelian subgroup of the group  $G$  has finite rank [in the sense of Prüfer].

Then  $G$  is  $\left\{ \begin{smallmatrix} \text{hypercentral} \\ \text{supersoluble} \end{smallmatrix} \right\}$  if, and only if,  $G$  is locally

$\left\{ \begin{smallmatrix} \text{hypercentral} \\ \text{supersoluble} \end{smallmatrix} \right\}$  [B. Theorem]. This result answers in a way our question I; and we use it to obtain criteria for a group to be noetherian and  $\left\{ \begin{smallmatrix} \text{hypercentral} \\ \text{supersoluble} \end{smallmatrix} \right\}$ .

In sections D, E we consider problem II for hypercentrality and a discussion of problem II for supersolubility will be found intermittingly throughout the second half of our paper.

In sections **G, H, I** we discuss the following condition which plays the same rôle for supersolubility which is played for hypercentrality by the soc. normalizer condition:

If  $X$  is a subgroup of  $G$ , equal to its normalizer and different from  $G$ , then there exists an element  $t$  in  $G$  with

$$X \subset \{X, t\} \text{ and } \{t^X\} \subseteq X_{\{X, t\}}\{t\}.$$

Here  $X_{\{X, t\}}$  is the product of all normal subgroups of  $\{X, t\}$  which are part of  $X$ .

#### NOTATIONS

$$x \circ y = x^{-1}y^{-1}xy$$

$A \circ B$  = subgroup, generated by the elements  $a \circ b$  with  $a$  in  $A$  and  $b$  in  $B$ .

$$G' = G \circ G$$

$$G^{(i+1)} = [G^{(i)}]'$$

$cS$  = centralizer of the subset  $S$  of the group  $G$

$nU$  = normalizer of the subgroup  $U$  of the group  $G$

$zG$  = center of the group  $G$

$z_i G$  is defined inductively by  $z_0 G = 1$ ,  $z_{i+1} G / z_i G = z[G / z_i G]$

$S_G$  = core of the subgroup  $S$  of  $G$  = product of all the normal subgroups of  $G$ , contained in  $S$  = intersection of all the subgroups, conjugate to  $S$  in  $G$ .

Factor of the group  $G$  = epimorphic image of a subgroup of  $G$

$A \subset B := A$  is a subgroup of  $B$  and  $A \neq B$

$o(\dots)$  = order of ...

$\{\dots\}$  = subgroup generated by ...

Locally  $e$  = every finitely generated subgroup meets requirement  $e$

$p$ -element = element of order a power of the prime  $p$

$p$ -group = group all of whose elements are  $p$ -elements

primary group =  $p$ -group for suitable  $p$

**A.** In this section we are going to collect a number of frequently used basic concepts and known results.

*Groups of finite rank:* Following Prüfer we say that the group  $G$  is of finite rank if there exists a positive integer  $n$  such that every finitely generated subgroup of  $G$  may be generated by  $n$  [or less] elements.

*The Components of an Abelian group:* If  $A$  is an abelian group, then the totality  $t_p A$  of elements of order a power of the prime  $p$  is a characteristic subgroup of  $A$ , its  $p$ -component. The direct product  $tA$  of all the  $t_p A$  is the *torsion subgroup* of  $A$ ; and  $t_0 A = A/tA$  is a torsionfree group, the *0-component* of  $A$ .

It is well known and easily verified that the following properties of the abelian group  $A$  are equivalent [see FUCHS]:

- (i) Every component of  $A$  is of finite rank.
- (ii) Every primary and every torsionfree subgroup of  $A$  is of finite rank.
- (iii) Every elementary primary factor of  $A$  is finite.

*Groups of finite Abelian subgroup rank* are groups whose abelian subgroups meet the above equivalent properties (i)–(iii).

*Groups of finite Abelian rank* are groups all of whose epimorphic images are of finite abelian subgroup rank.

It is clear that each of the following conditions is sufficient for the group  $G$  to be of finite abelian rank:

$G$  is noetherian;  $G$  is artinian;  $G$  is of finite rank.

*Hypercentrality of the group  $G$*  may be characterized by each of the following three equivalent properties of  $G$ :

- (i)  $\mathfrak{z}H \neq 1$  for every epimorphic image  $H \neq 1$  of  $G$ .
- (ii) If  $N \neq 1$  is a normal subgroup of the epimorphic image  $H$  of  $G$ , then  $N \cap \mathfrak{z}H \neq 1$ .
- (iii)  $\mathfrak{z}(G/C) \neq 1$  for every characteristic subgroup  $C \neq G$  of  $G$ .

See BAER [10; p. 17, Lemma 3.2]

It is well known that finitely generated hypercentral groups are noetherian—see BAER [2; p. 322, Satz 1]—and consequently nilpotent of finite class. Thus local hypercentrality and local nilpotency are equivalent properties.

If the group  $G$  is noetherian and hypercentral, then the set  $tG$  of elements of positive order in  $G$  is a finite and nilpotent characteristic subgroup of  $G$  with torsionfree  $G/tG$ ; see BAER [1; p. 207, Corollary]. It follows that every locally hypercentral group  $L$  has the following properties:

The totality  $L_p$  of elements of order a power of the prime  $p$  is a characteristic subgroup of  $L$  and the direct product of these primary components  $L_p$  of  $L$  is the torsion subgroup  $tL$  of  $L$  with torsionfree  $L/tL$ .

$\mathfrak{h}G$  = hypercenter of  $G$  = intersection of all normal subgroups  $X$  of  $G$  with  $\mathfrak{z}(G/X) = 1$ .

It has the following fundamental properties:

- (a)  $1 = \mathfrak{h}[G/\mathfrak{h}G] = \mathfrak{z}[G/\mathfrak{h}G]$
- (b) The subgroup  $S$  of  $G$  is hypercentral if, and only if,  $S \mathfrak{h}G/\mathfrak{h}G$  is hypercentral.

See BAER [1; p. 176/177].

*The normalizer condition* is satisfied by the group  $G$ , if  $S \subset nS$  for every subgroup  $S \neq G$  of  $G$ .

A subgroup  $S$  of  $G$  is termed *accessible*, if there exist subgroups  $S_\sigma [0 \leq \sigma \leq \beta]$  of  $G$  such that

$S = S_0$  and  $S_\beta = G$ ,

$S_\sigma$  is a normal subgroup of  $S_{\sigma+1}$  for  $0 \leq \sigma < \beta$ ,

$S_\lambda = \bigcup_{\sigma < \lambda} S_\sigma$  for limit ordinals  $\lambda \leq \beta$ .

It is almost obvious that the normalizer condition is satisfied by  $G$  if, and only if, every subgroup of  $G$  is accessible; and this implies that the normalizer condition is inherited by subgroups and epimorphic images.

(I) The normalizer condition is satisfied by every hypercentral group.

See KUROSH [II, p. 219].

An element  $a$  of  $G$  is termed an accessible element of  $G$ , if  $\{a\}$  is an accessible subgroup of  $G$ .

(II) If the normalizer condition is satisfied by  $G$ , then every element of  $G$  is accessible.

Obvious.

(III) If  $G$  is generated by its accessible elements, then every finitely generated subgroup of  $G$  is accessible and  $G$  is locally hypercentral.

See BAER [11; p. 57, Satz 3.3, p. 58, Zusatz 3.4, p. 59, Zusatz 3.6].

*Supersolubility of the group  $G$*  may be characterized by each of the following three equivalent properties of  $G$ :

- (i) Every epimorphic image, not 1, of  $G$  possesses a cyclic normal subgroup, not 1.
- (ii) If  $N \neq 1$  is a normal subgroup of the epimorphic image  $H$  of  $G$ , then there exists a cyclic normal subgroup  $Z$  of  $H$  with  $1 \subset Z \subseteq N$ .
- (iii) If  $C$  is a characteristic subgroup of  $G$  and  $G/C \neq 1$ , then there exists a cyclic normal subgroup, not 1, of  $G/C$ .

See BAER [3; p. 17, Lemma 2].

Subgroups and epimorphic images of supersoluble groups are supersoluble; see BAER [3; p. 16, Lemma 1].

Finitely generated supersoluble groups are noetherian; see BAER [3; p. 26, Theorem 1].

In analogy to the hypercenter of a group we define the subgroup  $jG$  of  $G$  as the intersection of all the normal subgroups  $X$  of  $G$  with the property:

- (j) 1 is the only cyclic normal subgroup of  $G/X$ .

This is a well determined characteristic subgroup of  $G$ , since  $X=G$  meets requirement (j). It is the smallest normal subgroup of  $G$ , meeting requirement (j). To prove this consider a normal subgroup  $K$  of  $G$  with  $jG \subseteq K$  and cyclic  $K/jG$ . If the normal subgroup  $X$  of  $G$  meets requirement (j), then  $jG \subseteq K \cap X$  and  $KX/X \cong K/(K \cap X)$  is cyclic as an epimorphic image of  $K/jG$ . Hence  $KX/X = 1$  so that  $K \subseteq X$  for every  $X$ , satisfying (j). This implies  $K \subseteq jG \subseteq K$  so that  $K = jG$  and  $jG$  meets requirement (j).

**B. Lemma:** *If the hypercentral group  $G$  is of finite abelian subgroup rank, then*

- (a) every primary subgroup of  $G$  is artinian,
- (b) every torsionfree epimorphic image of  $G$  is of finite rank and nilpotent of finite class and
- (c) every factor of  $G$  is of finite abelian rank.

Proof: Finitely many elements of finite order in the hypercentral group  $G$  generate a finite nilpotent subgroup of  $G$ ; see, for instance, BAER [1; p. 207, Corollary]. The set  $tG$  of all the elements of finite order in  $G$  is consequently a characteristic subgroup of  $G$ ; and it is the direct product of its primary components  $t_p G$ . Naturally  $G/tG$  is torsionfree.

If  $P$  is a  $p$ -subgroup of  $G$ , then  $P$  is hypercentral. If furthermore  $A$  is an abelian subgroup of  $P$ , then the set  $\bar{A}$  of all the elements  $a$  in  $A$  with  $a^p = 1$  is an elementary abelian  $p$ -subgroup of  $A$  and  $G$ . Consequently  $\bar{A}$  is of finite rank, implying the finiteness of  $\bar{A}$ . Hence  $A$  is artinian; see FUCHS [p. 68, 19]. Thus condition (3) of BAER [5; p. 530, Proposition] is satisfied by  $P$ , proving that  $P$  is artinian. This proves (a).

Application of MAL'CEV [p. 577, Theorem 5] shows that

(\*)  $G^* = G/tG$  is nilpotent of finite class and that

(\*\*) there exists a finite chain of normal subgroups  $N_j$  of  $G^*$  with

$$1 = N_0, N_i \subseteq N_{i+1}, N_k = G^*,$$

and every abelian subgroup of  $N_{i+1}/N_i$  is of finite rank.

[An even sharper form of the last property is assured by Mal'cev's Theorem].

We consider the ascending central chain  ${}_i G^*$  of  $G^*$  [which is defined by the rules:  ${}_0 G^* = 1$ ,  ${}_{i+1} G^* / {}_i G^* = {}_i(G^* / {}_i G^*)$ ]. It is a consequence of (\*) that

$$G^* = {}_c G^* \text{ for some } c.$$

Since  $G^*$  is torsionfree and nilpotent of finite class, every  $G^* / {}_i G^*$  is torsionfree, see BAER [1; p. 200, Corollary 1]. Every factor of  $G^*$  meets requirement (\*\*). Thus  ${}_i(G^* / {}_i G^*)$  is a torsionfree abelian group, meeting requirement (\*\*); and as such it is of finite rank. Hence every

$${}_{i+1} G^* / {}_i G^* \text{ is of finite rank.}$$

But an extension of a group of finite rank by a group of finite rank is easily seen to be a group of finite rank; and now it follows by complete induction that  $G^*$  itself is of finite rank, and this implies (b).

If the elementary abelian  $p$ -group  $A$  is a factor of  $G$ , then there exists an epimorphism  $\sigma$  of a subgroup  $S$  of  $G$  upon  $A$ . Since the hypotheses imposed upon  $G$  are likewise satisfied by the subgroup  $S$  of  $G$ , the conclusions (a), (b) already verified, are also true of  $S$ . Since  $A$  is a  $p$ -group, so is every subgroup of  $A$  and we have

$$(tS)^\sigma = (t_p S)^\sigma = B.$$

But  $t_p S$  is artinian by (a). Hence  $B$  is an artinian elementary abelian group [as a subgroup of  $A$ ] and as such  $B$  is finite. Next  $A/B = S^\sigma / tS^\sigma$  is an epimorphic image of  $S/tS$ . Since the latter group is of finite rank by (b), so is the former; and  $A/B$  is finite as an elementary abelian  $p$ -group of finite rank. Hence  $A$  itself is finite; and we have verified the following intermediate result:

(c\*) Every elementary abelian  $p$ -factor of  $G$  is finite.

Consider now an abelian  $p$ -factor  $P$  of  $G$ . It is an almost immediate consequence of (c\*) that  $P$  possesses but a finite number of elements of order  $p$ ; and this shows that  $P$  is artinian and of finite rank; see FUCHS [p. 65, Theorem 19.2 and p. 68, 19].

Consider next a torsionfree abelian factor  $F$  of  $G$ . Then  $F$  contains a free abelian subgroup  $E$  with  $F/E$  a torsiongroup. If  $p$  is any prime, then  $E/E^p$  is an elementary abelian  $p$ -factor of  $G$ . As we have shown just now this implies the finiteness of  $E/E^p$ . But then  $E$  is free abelian of finite rank, proving the finiteness of the rank of  $F$  and the validity of (c).

**Remark:** It is easy to construct abelian torsiongroups whose primary components are of finite rank, though the group itself is not of finite rank. Thus it is quite impossible to show that hypercentral groups of finite abelian subgroup rank are of finite rank. If  $A$  is a group of Prüfer's type  $2^\infty$  and if  $G$  arises from  $A$  by adjoining to  $A$  an element  $b$  subject to the relations

$$(ab)^2 = 1 \text{ for every } a \text{ in } A,$$

then  $G$  is hypercentral and of rank 2, but not of finite class. Thus it is quite impossible to show that the groups, considered in our Lemma, are in general of finite class.

**Theorem:** *The following properties of the group  $G$  of finite abelian subgroup rank are equivalent:*

- (i)  $G$  is hypercentral.
- (ii) The normalizer property is satisfied by  $G$ .
- (iii)  $G$  is generated by its accessible elements.
- (iv)  $G$  is locally hypercentral.
- (v) Finitely generated subgroups of  $G$  are noetherian and nilpotent of finite class.

**Remark:** This theorem generalizes in a way some results of MAL'CEV [p. 577, Theorem 5 and p. 579, Corollary]. It should be noted, however, that Mal'cev's Theorem 5 is used in the proof of our theorem. See also the Corollary below.

**Proof:** We have pointed out in section A that (i) implies (ii), that (ii) implies (iii), that (iii) implies (iv) and that (iv) and (v) are equivalent.

Assume now the validity of (v). We recall that the hypercenter  $\mathfrak{h}G$  is a hypercentral characteristic subgroup of  $G$  with  $\mathfrak{z}(G/\mathfrak{h}G)=1$  and the following further property:

The subgroup  $S$  of  $G$  with  $\mathfrak{h}G \subseteq S$  is hypercentral if, and only if,  $S/\mathfrak{h}G$  is hypercentral.

See BAER [1; p. 177].

Consider now a subgroup  $S$  of  $G$  with  $S' \subseteq \mathfrak{h}G \subseteq S$ . Then  $S$  is hypercentral and of finite abelian subgroup rank. It is a consequence of our Lemma, (c) that  $S$  is of finite abelian rank. Hence every component of the abelian group  $S/\mathfrak{h}G$  is of finite rank; and thus we have shown:

(+)  $H=G/\mathfrak{h}G$  is of finite abelian subgroup rank.

Assume by way of contradiction that  $H \neq 1$ . If  $H$  were torsionfree, then  $H$  would be a locally nilpotent group all of whose abelian subgroups are torsionfree of finite rank. Application of a Theorem of MAL'CEV [p. 577, Theorem 5] shows that  $H$  is nilpotent of finite class. This implies  $\mathfrak{z}H \neq 1$ , since  $H \neq 1$ , a contradiction. Hence  $H$  is not torsionfree. Consequently there exists a prime  $p$  such that  $H$  contains elements of order  $p$ . The group  $H$  contains elementary abelian  $p$ -groups and application of the Maximum Principle of Set Theory shows the existence of a maximal elementary abelian  $p$ -subgroup  $M$  of  $H$ . Since  $H$  contains elements of order  $p$ , we conclude that  $M \neq 1$ ; and we deduce from (+) that  $M$  is of finite rank and hence finite.

If  $F$  is a finitely generated subgroup of  $H$ , then  $\{M, F\}$  is a finitely generated subgroup of  $H$ . As such  $\{M, F\}$  is an epimorphic image of a finitely generated subgroup of  $G$ ; and we deduce from (v) that  $\{M, F\}$  is noetherian of finite class. The totality  $t_p\{M, F\}$  of all  $p$ -elements in  $\{M, F\}$  is, as we pointed out in A, a finite characteristic  $p$ -subgroup of  $\{M, F\}$ . Naturally  $1 \subset M \subseteq t_p\{M, F\}$  and this implies

$$t_p\{M, F\} \cap \mathfrak{z}\{M, F\} \neq 1$$

by a result recorded in A. Consequently there exists an element  $x$  of order  $p$  in  $\mathfrak{z}\{M, F\}$ . Clearly  $\{M, x\}$  is an elementary abelian  $p$ -subgroup of  $H$ ; and we deduce  $M = \{M, x\}$  from the maximality of  $M$ . Thus  $x$  belongs to  $M \cap cF$ , and we have shown that

(++)  $M \cap cF \neq 1$  for every finitely generated subgroup  $F$  of  $H$ .

Since  $M$  is finite, there exists among the finitely generated subgroups of  $H$  one, say  $N$ , with  $M \cap cN$  of minimal order. If  $x$  is any element in  $H$ , then  $\{N, x\}$  is a finitely generated subgroup of  $H$ . Naturally  $c\{N, x\} \subseteq cN$  so that  $M \cap c\{N, x\} \subseteq M \cap cN$ . Because of the minimality of  $M \cap cN$  we find

$$M \cap cN = M \cap c\{N, x\} \subseteq cx \text{ for every } x \text{ in } H.$$

It follows that

$$M \cap cN \subseteq {}_3H$$

and this implies  ${}_3H \neq 1$  by  $(++)$ . This contradiction shows that  $H = 1$ . Hence  $G = {}_1G$  is hypercentral; and we have completed the proof of the equivalence of conditions (i)–(v).

**Corollary:** *The group  $G$  is noetherian and hypercentral if, and only if,  $G$  is locally hypercentral and every abelian subgroup of  $G$  is finitely generated.*

This result is due to MAL'CEV [p. 579, Corollary].

**Proof:** If  $G$  is locally hypercentral and the abelian subgroups of  $G$  are finitely generated, then  $G$  is by B, Theorem hypercentral. Thus  $G$  is soluble in the weak sense that every epimorphic image, not 1, of  $G$  possesses an abelian normal subgroup, not 1. Hence we may apply a generalization of a theorem of Mal'cev—see BAER [4; p. 173, Hauptsatz 4]—to show that  $G$  is noetherian.

**C. Theorem:** *The following properties of the group  $G$  are equivalent.*

- (i)  $G$  is hypercentral.
- (ii)  $\left\{ \begin{array}{l} \text{(a) } G \text{ is locally hypercentral.} \\ \text{(b) Every epimorphic image, not 1, of } G \text{ possesses a finitely} \\ \text{generated normal subgroup, not 1.} \end{array} \right.$
- (iii)  $\left\{ \begin{array}{l} \text{(a) } G \text{ is locally hypercentral.} \\ \text{(b) Every epimorphic image, not 1, of } G \text{ possesses a normal sub-} \\ \text{group, not 1, of finite abelian subgroup rank.} \end{array} \right.$
- (iv)  $\left\{ \begin{array}{l} \text{(a) } G \text{ is locally hypercentral.} \\ \text{(b) Every epimorphic image, not 1, of } G \text{ possesses an abelian} \\ \text{normal subgroup, not 1, whose components are of finite rank.} \end{array} \right.$
- (v)  $\left\{ \begin{array}{l} \text{(a) Every finite factor of } G \text{ is nilpotent.} \\ \text{(b) Every epimorphic image } H \neq 1 \text{ of } G \text{ possesses an abelian normal} \\ \text{subgroup } A \neq 1 \text{ such that } H/cA \text{ is finitely generated.} \end{array} \right.$
- (vi)  $\left\{ \begin{array}{l} \text{(a) Every finite factor of } G \text{ is nilpotent.} \\ \text{(b) Every epimorphic image, not 1, of } G, \text{ possesses a finitely} \\ \text{generated, abelian, normal subgroup, not 1.} \end{array} \right.$

**Proof:** If  $G$  is hypercentral, then every subgroup of  $G$  is hypercentral so that  $G$  is locally hypercentral. If furthermore  $H \neq 1$  is an epimorphic image of  $G$ , then  ${}_3H \neq 1$  and the cyclic subgroups, not 1, of  ${}_3H$  are normal subgroups of  $H$ . Thus (ii) is a consequence of (i).

If (ii) is satisfied by  $G$  and if  $H \neq 1$  is an epimorphic image of  $G$ , then there exists a finitely generated normal subgroup  $N \neq 1$  of  $H$ . It is a consequence of (ii. a) that  $N$  is hypercentral and noetherian—see A.



Hence  ${}_3N \neq 1$  is a finitely generated abelian normal subgroup of  $H$  [as a characteristic subgroup of a normal subgroup]. Thus (iii) is a consequence of (ii).

Assume next the validity of (iii). If  $H \neq 1$  is an epimorphic image of  $G$ , then there exists a normal subgroup  $N \neq 1$  of  $H$  such that  $N$  is of finite abelian subgroup rank. Since  $H$  is locally hypercentral, so is  $N$ . Application of **B**, Theorem shows that  $N$  is hypercentral. Hence  ${}_3N \neq 1$  is an abelian characteristic subgroup of  $N$ . It follows that  ${}_3N$  is an abelian normal subgroup, not 1, of  $H$  whose components are of finite rank. Thus (iv) is a consequence of (iii).

Assume now the validity of (iv) and consider an epimorphic image  $H \neq 1$  of  $G$ . From (iv. a) we deduce the local hypercentrality of  $H$ ; and from (iv. b) we deduce the existence of an abelian normal subgroup  $N \neq 1$  of  $H$  whose components are of finite rank.

*Case 1:  $N$  is not torsionfree.*

Then there exists a prime  $p$  such that  $N$  contains elements of order  $p$ . The set  $P$  of all elements  $x$  in  $N$  with  $x^p = 1$  is a characteristic subgroup of  $N$  and hence a normal subgroup of  $H$ . Furthermore  $P$  is an elementary abelian  $p$ -group, not 1. Since the components of  $N$  are of finite rank,  $P$  is of finite rank and hence finite. The centralizer  $c_P$  of  $P$  is a normal subgroup of  $H$ ; and  $H/c_P$  is essentially the same as the group of automorphisms, induced in  $P$  by  $H$ . Since  $P$  is finite, so is  $H/c_P$ . Consequently there exists a finitely generated subgroup  $F$  of  $H$  with  $P \subseteq F$  and  $H = Fc_P$ . Since  $H$  is locally hypercentral,  $F$  is hypercentral and  $P \neq 1$  is a normal subgroup of  $F$ . Consequently  $1 \neq P \cap {}_3F$ —see **A**. Since  $P \cap {}_3F$  is centralized both by  $c_P$  and  $F$ , it is centralized by  $Fc_P = H$  so that  $1 \subset P \cap {}_3F \subseteq {}_3H$ .

*Case 2:  $N$  is torsionfree.*

Then  $N$  is a torsionfree abelian group of finite rank and  $N \neq 1$ . If  $X$  is a finitely generated subgroup of  $H$ , then  $X$  is hypercentral and—by **A**—noetherian. The subgroup  $NX$  of  $H$  is locally hypercentral—with  $H$ —and it is an extension of its abelian normal subgroup  $N$  by the finitely generated hypercentral group  $NX/N \cong X/(N \cap X)$ . But we showed elsewhere—see **BAER** [2; p. 310, Satz 1]—that such a group is hypercentral. Since  $N \neq 1$  is a normal subgroup of  $NX$ , it follows that  $1 \neq N \cap {}_3(NX)$ . Consider now an element  $s$  in  $N$  such that  $s^i$  belongs to  $N \cap {}_3(NX)$  for some integer  $i \neq 0$ . If  $x$  is an element in  $NX$ , then  $s$  and  $s^x$  belong to the abelian normal subgroup  $N$ . Hence

$$[s^x s^{-1}]^i = (s^x)^i s^{-i} = (s^i)^x s^{-i} = 1;$$

and this implies  $s^x s^{-1} = 1$ , since  $N$  is torsionfree. Thus  $s$  itself belongs to  $N \cap {}_3(NX)$  and we have shown:

- (-) If  $X$  is a finitely generated subgroup of  $H$ , then  $N \cap {}_3(NX) \neq 1$  and  $N/[N \cap {}_3(NX)]$  is torsionfree.

Since  $N$  is of finite rank, so is  $N \cap \mathfrak{z}(NX)$  for every finitely generated subgroup  $X$  of  $H$ . The rank  $\mathfrak{r}[N \cap \mathfrak{z}(NX)]$  is the maximal number of independent elements in the torsionfree abelian group  $N \cap \mathfrak{z}(NX)$ . This number is a finite positive integer by (+); and among all these finite positive integers there exists a minimal one, say  $m$ . There exists a finitely generated subgroup  $F$  of  $H$  with  $m = \mathfrak{r}[N \cap \mathfrak{z}(NF)]$ .

Consider now an element  $h$  in  $H$ . Then  $\{F, h\}$  is a finitely generated subgroup of  $H$  and

$$m \leq \mathfrak{r}[N \cap \mathfrak{z}\{N, F, h\}]$$

by the choice of  $m$ . If  $s$  is an element in  $N \cap \mathfrak{z}\{N, F, h\}$ , then  $s$  is likewise centralized by  $NF$  and belongs consequently to  $N \cap \mathfrak{z}(NF)$ . Thus we have shown that

$$N \cap \mathfrak{z}\{N, F, h\} \subseteq N \cap \mathfrak{z}(NF).$$

We conclude

$$\mathfrak{r}[N \cap \mathfrak{z}\{N, F, h\}] = \mathfrak{r}[N \cap \mathfrak{z}(NF)] = m.$$

Since  $[N \cap \mathfrak{z}(NF)]/[N \cap \mathfrak{z}\{N, F, h\}]$  is torsionfree by (+), and since  $m$  is finite, it follows that

$$N \cap \mathfrak{z}\{N, F, h\} = N \cap \mathfrak{z}(NF).$$

Thus  $N \cap \mathfrak{z}(NF)$  is centralized by every element  $h$  in  $H$ ; and we deduce from (+) that

$$1 \subset N \cap \mathfrak{z}(NF) \subseteq \mathfrak{z}H.$$

Now we have shown  $\mathfrak{z}H \neq 1$  in either case so that  $G$  is hypercentral, proving the equivalence of conditions (i)–(iv).

It is quite obvious that hypercentral groups meet requirement (vi). If (vi) is satisfied by  $G$ , then (v. a) is clearly satisfied by  $G$ . Furthermore  $G$  and all its epimorphic images have the following weak solubility property:

- (+) Every epimorphic image, not 1, possesses an abelian normal subgroup, not 1.

If  $H \neq 1$  is an epimorphic image of  $G$ , then there exists a finitely generated abelian normal subgroup  $A \neq 1$  of  $H$ . The group of automorphisms, induced in  $A$  by  $H$ , is essentially the same as  $H/cA$ . If a group of automorphisms of a finitely generated abelian group meets the solubility requirement (+), then this group of automorphisms is known to be noetherian; see BAER [4; p. 171, Satz 2]. Consequently  $H/cA$  is noetherian, showing that (v. b) is a consequence of (vi. b). Hence (v) is a consequence of (vi).

Before deriving hypercentrality from (v) we treat a special case. Suppose that the following conditions are satisfied by the group  $F$ :

- (a) Every finite factor of  $F$  is nilpotent.
- (b) Every epimorphic image  $H \neq 1$  of  $F$  possesses an abelian normal subgroup  $A \neq 1$ .
- (c)  $F$  is finitely generated.

Assume by way of contradiction that

- (d)  $F$  is not hypercentral.

Application of BAER [12; p. 410, Lemma 4] shows the existence of an epimorphic image  $H$  of  $F$ , meeting the following requirements:

- (e)  $H$  is not hypercentral, but every proper epimorphic image of  $H$  is hypercentral.

By (b) there exists an abelian normal subgroup  $A \neq 1$  of  $H$  [since  $H \neq 1$  by (e)]. We deduce from (c) that  $H$  and  $H/A$  are finitely generated; and it is a consequence of (e) that  $H/A$  is hypercentral and hence noetherian — see A. — Since  $A$  is abelian and  $H/A$  hypercentral and noetherian, we may, because of (a), apply BAER [2; p. 310, Satz 1]. It follows that  $H$  is hypercentral, contradicting (e). Thus our assumption (d) has led to a contradiction and we have shown the following intermediate result:

- (v\*) If the group  $F$  is finitely generated, if every finite factor of  $F$  is nilpotent, and if every epimorphic image, not 1, of  $F$  possesses an abelian normal subgroup, not 1, then  $F$  is hypercentral [and noetherian].

Suppose now that condition (v) is satisfied by the group  $G$  and consider an epimorphic image  $H \neq 1$  of  $G$ . By (v. b) there exists an abelian normal subgroup  $A \neq 1$  of  $H$  with finitely generated  $H/cA$ . Consequently there exists a finitely generated subgroup  $F$  of  $H$  with  $H = FcA$ ; and we may assume without loss in generality that  $F \cap A \neq 1$ . It is a consequence of (v. a) that every finite factor of  $F$  [as a factor of  $G$ ] is nilpotent. We deduce from (v. b) that every epimorphic image, not 1, of  $G$  possesses an abelian normal subgroup, not 1. This property is inherited by subgroups. Hence (v\*) may be applied upon  $F$ . Thus  $F$  is hypercentral. Since  $F \cap A$  is a normal subgroup, not 1, of  $F$ , we find that

$$1 \neq (F \cap A) \cap {}_3F = A \cap {}_3F;$$

see A. Clearly  $A \cap {}_3F$  is centralized by  $F$  and  $cA$  and hence by  $FcA = H$  so that

$$1 \subset A \cap {}_3F \subseteq {}_3H,$$

proving the hypercentrality of  $G$  and the equivalence of conditions (i)–(vi).

*Notes on the relative strength of conditions (i)–(vi):*

1. Because of the results, quoted in section A, it is possible to substitute everywhere for the requirement of local hypercentrality such stronger

conditions as the normalizer condition or the condition that  $G$  be generated by its accessible elements.

2. If a group is locally hypercentral, then it is almost obvious that all its finite factors are nilpotent. The converse is false, since Golod-Safarevic have constructed an infinite, finitely generated  $p$ -group. Its finite factors are finite  $p$ -groups and hence nilpotent. It is not hypercentral, since finitely generated hypercentral torsion groups are finite, see BAER [1; p. 201, Corollary].

3. It has been shown in the course of the proof of our theorem that (v. b) is a consequence of (vi. b), though the theorems applied in this proof are not at all trivial. The converse is false, as may be seen from the following often used example:

Denote by  $A$  a free abelian group of countably infinite rank. Then there exists a basis  $b(i)$  with  $i=0, \pm 1, \pm 2, \dots$  of  $A$ . Consequently there exists a group  $G$  which arises from  $A$  by adjoining an element  $b$  subject to the relations:

$$b^{-1}b(i)b = b(i+1) \text{ for every } i.$$

Clearly  $G$  is soluble in the very strict sense that  $G''=1$ . Furthermore  $G$  is generated by the elements  $b$  and  $b(0)$ . Hence condition (v. b) is satisfied by  $G$ . But 1 is the only finitely generated abelian normal subgroup of  $G$ . Hence (vi. b) is not satisfied by  $G$ .

4. It is clear that condition (iv. b) is considerably weaker than condition (vi. b).

*Corollary: The finitely generated group  $G$  is hypercentral if, and only if,*

- (a) *every finite factor of  $G$  is nilpotent and*
- (b) *every epimorphic image, not 1, of  $G$  possesses an abelian normal subgroup, not 1.*

This is easily deduced from our preceding theorem, since the present conditions imply condition (v) of that theorem.

**D.** In this section we are going to derive criteria for a product of hypercentral normal subgroups to be hypercentral. This is, in general, not the case, as may be seen from the following well known and often discussed

**Example D.1:** There exists a group  $G$  with normal subgroup  $N$ , meeting the following requirements.

$N$  and  $G/N$  are countably infinite elementary abelian  $p$ -groups and  $\mathfrak{z}G=1$ .

See e.g. BAER [11; p. 69, 6.1].

If  $g$  is any element in  $G$ , then  $N\{g\}$  is a normal  $p$ -subgroup of  $G$  whose

class is finite [not exceeding  $p+1$ ]. Thus  $G$  is the product of hypercentral normal subgroups—they are of finite class—though  $G$  is certainly not hypercentral.

**Proposition D.2:** *Products of finitely many hypercentral normal subgroups are hypercentral.*

**Proof:** It suffices to show that every product of two hypercentral normal subgroups is hypercentral. If  $G$  is the product of two hypercentral normal subgroups, so is every epimorphic image  $H \neq 1$  of  $G$ . Hence  $H = AB$  where  $A$  and  $B$  are hypercentral normal subgroups of  $H$ . If  $A = 1$ , then  $H = B$  is hypercentral implying  ${}_3H \neq 1$ ; and likewise  ${}_3H \neq 1$ , if  $B = 1$ . Thus we assume  $A \neq 1$  and  $B \neq 1$ . Because of the hypercentrality of  $A$  we have  ${}_3A \neq 1$ ; and the characteristic subgroup  ${}_3A$  of  $A$  is a normal subgroup of  $H$ . If firstly  ${}_3A \cap B = 1$ , then  ${}_3A$  is centralized by  $A$  and  $B$  and hence by  $H$  so that  $1 \subset {}_3A \subseteq {}_3H$ . If secondly  $1 \neq B \cap {}_3A$ , then  $B \cap {}_3A$  is a normal subgroup, not 1, of the hypercentral group  $B$  so that

$$1 \subset (B \cap {}_3A) \cap {}_3B = {}_3A \cap {}_3B \subseteq {}_3H.$$

Thus we have shown  ${}_3H \neq 1$  in all cases, proving the hypercentrality of  $G$ .

We recall that the *Hirsch-Plotkin radical*  $\mathfrak{H}\mathfrak{P}G$  of a group  $G$  is the product of all the locally hypercentral normal subgroups of  $G$  and that  $\mathfrak{H}\mathfrak{P}G$  is always a locally hypercentral characteristic subgroup of  $G$ .

**Proposition D.3:** *If  $G$  is of finite abelian subgroup rank, then  $\mathfrak{H}\mathfrak{P}G$  is hypercentral and*

$$\begin{aligned} \mathfrak{H}\mathfrak{P}G &= \text{set of all accessible elements of } G \\ &= \text{compositum of all accessible hypercentral subgroups of } G. \end{aligned}$$

**Proof:**  $\mathfrak{H}\mathfrak{P}G$  is [with  $G$ ] of finite abelian subgroup rank. Apply the local hypercentrality of  $\mathfrak{H}\mathfrak{P}G$  and **B**, Theorem to show the hypercentrality of  $\mathfrak{H}\mathfrak{P}G$ .

The normalizer condition is satisfied by every hypercentral group. Hence every element in  $\mathfrak{H}\mathfrak{P}G$  is an accessible element of  $\mathfrak{H}\mathfrak{P}G$  and consequently of  $G$ . On the other hand it is known that the set  $A$  of all accessible elements of  $G$  is a locally hypercentral characteristic subgroup of  $G$ ; see BAER [11; p. 57, Satz 3.3 and p. 59, Zusatz 3.6]. Thus  $A$  is part of the Hirsch-Plotkin radical of  $G$ , proving  $A = \mathfrak{H}\mathfrak{P}G$ .

If  $X$  is an accessible hypercentral subgroup of  $G$ , then every element in  $X$  is an accessible element of  $X$ —see **A**—and consequently of  $G$ . Hence  $X \subseteq A = \mathfrak{H}\mathfrak{P}G$  so that the Hirsch-Plotkin radical is as a hypercentral characteristic subgroup of  $G$  the compositum of all the hypercentral accessible subgroups of  $G$ .

**E.** Denote by  $\mathfrak{C}$  a non-vacuous class of groups which contains with any group all its subgroups and isomorphic images. Denote by  $\mathfrak{f}$  a function

assigning to every group  $G$  in  $\mathfrak{C}$  a subgroup  $\mathfrak{f}G$  of  $G$ , subject to the following rules:

- (I)  $\mathfrak{f}(G^\sigma) = (\mathfrak{f}G)^\sigma$  for every group  $G$  in  $\mathfrak{C}$  and every isomorphism  $\sigma$  of  $G$ .
- (II)  $\mathfrak{f}X \subseteq \mathfrak{f}Y$  for  $X \subseteq Y$  in  $\mathfrak{C}$ .
- (III) To every element  $x$  in  $\mathfrak{f}G$  there exists a finitely generated subgroup  $X$  of  $G$  with  $x$  in  $\mathfrak{f}X$ .

An obvious example of such a function is the one defined by  $\mathfrak{f}G = G'$ .

We note that  $\mathfrak{f}G$  is because of (I) a characteristic subgroup of  $G$  and that because of (II) our condition (III) is equivalent with the following requirement:

- (III\*) The finitely generated subgroup  $F$  of  $G$  is part of  $\mathfrak{f}G$  if, and only if, there exists a finitely generated subgroup  $E$  of  $G$  with  $F \subseteq \mathfrak{f}E$ .

The following result will prove the key to our discussion of supersolubility.

**Proposition:** *Assume that the function  $\mathfrak{f}$  on  $\mathfrak{C}$  meets requirements (I)–(III) and that  $G$  is a group in  $\mathfrak{C}$ .*

- (a)  $\mathfrak{f}G$  is locally hypercentral if, and only if,  $\mathfrak{f}X$  is locally hypercentral for every finitely generated subgroup  $X$  of  $G$ .
- (b) If  $G$  is of finite abelian subgroup rank, then the following properties of  $G$  are equivalent.
  - (i).  $\mathfrak{f}G$  is hypercentral and of finite abelian rank.
  - (ii)  $\mathfrak{f}X$  is locally hypercentral for every finitely generated subgroup  $X$  of  $G$ .

**Proof:** If  $\mathfrak{f}G$  is locally hypercentral and  $X$  is a subgroup of  $G$ , then  $\mathfrak{f}X \subseteq \mathfrak{f}G$  by (II) so that  $\mathfrak{f}X$  too is locally hypercentral. If conversely  $\mathfrak{f}X$  is locally hypercentral for every finitely generated subgroup  $X$  of  $G$ , and if  $F$  is a finitely generated subgroup of  $\mathfrak{f}G$ , then we deduce from (III\*) the existence of a finitely generated subgroup  $E$  of  $G$  with  $F \subseteq \mathfrak{f}E$ . Since  $\mathfrak{f}E$  is by hypothesis locally hypercentral, its finitely generated subgroup  $F$  is hypercentral. This proves (a).

Assume in addition that  $G$  is of finite abelian subgroup rank. Then it is clear that (i) implies (ii). If (ii) is true, then we deduce from (a) that  $\mathfrak{f}G$  is locally hypercentral. Since  $\mathfrak{f}G$  is with  $G$  of finite abelian subgroup rank, we deduce from **B**, Theorem the hypercentrality of  $\mathfrak{f}G$ ; and it follows from **B**, Lemma that  $\mathfrak{f}G$  is of finite abelian rank.

**F. Theorem:** *The group  $G$  of finite abelian subgroup rank is supersoluble if, and only if,  $G$  is locally supersoluble.*

**Proof:** Since subgroups of supersoluble groups are supersoluble—see BAER [3; p. 16, Lemma 1]—supersoluble groups are locally supersoluble.

Assume conversely the local supersolubility of  $G$ . Let  $\mathfrak{C}$  be the class

of all groups and  $\{G = G'\}$ . Note that  $X'$  is hypercentral for every supersoluble group  $X$  by BAER [3; p. 21, Proposition 2]. Since  $G$  is of finite abelian subgroup rank, application of E, Proposition shows that

- (1)  $G'$  is hypercentral and of finite abelian rank.

Next we prove two special instances of what we intend to prove. In both cases our hypotheses are slightly weaker than those at our disposal.

- (2) If  $Y \neq 1$  is a finite normal subgroup of the locally supersoluble group  $X$ , then  $Y$  contains a cyclic normal subgroup, not 1, of  $X$ .

**Proof:** The centralizer  $cY$  is a normal subgroup of  $X$  and  $X/cY$  is essentially the same as the group of automorphisms, induced by  $X$  in  $Y$ . Since  $Y$  is finite, so is  $X/cY$ . Consequently there exists a finitely generated subgroup  $F$  of  $X$  with  $Y \subseteq F$  and  $X = FcY$ . By hypothesis  $F$  is supersoluble. The normal subgroup  $Y \neq 1$  of  $F$  contains therefore a cyclic normal subgroup  $Z \neq 1$  of  $F$ ; see BAER [3; p. 17, Lemma 2]. Since  $Z$  is normalized by  $F$  and centralized by  $cY$  [because of  $Z \subseteq Y$ ],  $Z$  is normalized by  $FcY = X$ . Thus  $Z$  is a cyclic normal subgroup of  $X$  with  $1 \subset Z \subseteq Y$ .

- (3) If the group  $X$  is locally supersoluble, and if  ${}_3X'$  is torsionfree of finite rank and different from 1, then there exists a cyclic normal subgroup  $Z$  of  $X$  with  $1 \subset Z \subseteq {}_3X'$ .

**Proof:** We note first that  ${}_3X'$  is a characteristic subgroup of  $X$  and that an abelian group of automorphisms is induced in  ${}_3X'$  by  $X$ .

If  $x$  is an element in  $X$  and  $A$  a subgroup of  ${}_3X'$ , then we term  $A$  an  $x$ -satisfying subgroup of  ${}_3X'$ , if there exists an integer  $e = e(x, A)$  with

$$e^2 = 1 \text{ and } ax = a^e \text{ for every } a \text{ in } A.$$

This implies that every subgroup of  $A$  is normalized by the element  $x$ .

Consider an element  $x$  in  $X$  and an  $x$ -satisfying subgroup  $A$  of  ${}_3X'$ . If  $y$  is some element in  $X$  and  $a$  belongs to  $A$ , then we recall that  $X$  induces an abelian group of automorphisms in  ${}_3X'$  so that

$$(ay)^x = (a^x)^y = (a^{e(x,A)})^y = (ay)^{e(x,A)}.$$

Naturally  $\{A^X\} \subseteq {}_3X'$ . If  $b$  is an element in  $\{A^X\}$ , then there exist finitely many elements  $a(i)$  in  $A$  and  $y(i)$  in  $X$  with

$$b = \prod_i a(i)^{y(i)}.$$

Application of what we had shown just now proves that

$$bx = \prod_i [a(i)^{y(i)}]^x = \prod_i [a(i)^{y(i)}]^{e(x,A)} = b^{e(x,A)},$$

since  ${}_3X'$  and its subgroup  $\{A^X\}$  are abelian. Thus we have shown:

- (a) If  $A$  is an  $x$ -satisfying subgroup of  ${}_3X'$ , then  $\{A^X\}$  is an  $x$ -satisfying subgroup of  ${}_3X'$ .

Consider an  $x$ -satisfying subgroup  $A$  of  ${}_3X'$  and denote by  $A^*$  the uniquely determined subgroup of  ${}_3X'$  with  $A \subseteq A^*$  and  $A^*/A$  the torsion subgroup of  ${}_3X'/A$ . If  $b$  is an element in  $A^*$ , then there exists a positive integer  $n$  such that  $b^n$  belongs to  $A$ . We note that  $b$  and  $b^x$  belong both to the abelian characteristic subgroup  ${}_3X'$  of  $X$ . Hence

$$\begin{aligned} [b^x b^{-e(x,A)}]^n &= (b^x)^n b^{-e(x,A)n} = (b^n)^x b^{-e(x,A)n} = \\ &= (b^n)^{e(x,A)} b^{-e(x,A)n} = 1. \end{aligned}$$

But  ${}_3X'$  is torsionfree so that  $b^x b^{-e(x,A)} = 1$  and

$$b^x = b^{e(x,A)} \text{ for every } b \text{ in } A^*.$$

Hence  $A^*$  too is an  $x$ -satisfying subgroup of  ${}_3X'$  proving that

(b) to every  $x$ -satisfying subgroup  $A$  of  ${}_3X'$  there exists an  $x$ -satisfying subgroup  $A^*$  of  ${}_3X'$  with  $A \subseteq A^*$  and torsionfree  ${}_3X'/A^*$ .

Now consider a normal subgroup  $S$  of  $X$  with  $1 \subset S \subseteq {}_3X'$  and an element  $x$  in  $X$ . There exists an element  $s \neq 1$  in  $S$  and the finitely generated subgroup  $\{s, x\}$  of  $X$  is by hypothesis supersoluble. Naturally  $S \cap \{s, x\}$  is a normal subgroup, not 1, of the supersoluble group  $\{s, x\}$ . Consequently there exists a cyclic normal subgroup  $Z$  of  $\{s, x\}$  with

$$1 \subset Z \subseteq S \cap \{s, x\} \subseteq {}_3X';$$

see BAER [3; p. 17, Lemma 2]. Since  ${}_3X'$  is torsionfree,  $Z$  is an infinite cyclic group which is normalized by  $x$ . Since the group of automorphisms of  $Z$  is cyclic of order 2, it follows that  $Z$  is  $x$ -satisfying. Thus we have shown the following fact:

(c) If  $S$  is a normal subgroup of  $X$  with  $1 \subset S \subseteq {}_3X'$  and  $x$  is an element in  $X$ , then  $S$  contains an  $x$ -satisfying subgroup, not 1, of  ${}_3X'$ .

Denote by  $\Sigma$  the set of all subgroups  $A$  with

$$1 \subset A = A^X \subseteq {}_3X' \text{ and torsionfree } {}_3X'/A.$$

Naturally  ${}_3X'$  belongs to  $\Sigma$ . Since  ${}_3X'$  is torsionfree of finite positive rank, so is every subgroup in  $\Sigma$ . Thus there exists among the subgroups in  $\Sigma$  one  $M$  of minimal rank.

If  $x$  is an element in  $X$ , then we denote by  $\Sigma_x$  the set of all  $x$ -satisfying subgroups  $A$  of  ${}_3X'$  with

$$1 \subset A \subseteq M \text{ and torsionfree } {}_3X'/A.$$

A combination of (c) and (b) shows that  $\Sigma_x$  is not vacuous. Since the ranks of all the groups in  $\Sigma_x$  are bounded by the finite positive rank of the torsionfree group  ${}_3X'$ , there exists among the groups in  $\Sigma_x$  one  $M_x$  of maximal rank. Clearly

$$M_x^X \subseteq M^X = M;$$

and it is a consequence of (a) that  $\{M_x^X\}$  is  $x$ -satisfying. By (b) there exists an  $x$ -satisfying subgroup  $V$  with

$$M_x \subseteq \{M_x^X\} \subseteq V \subseteq {}_3X' \text{ and torsionfree } {}_3X'/V.$$



Then  $M \cap V$  is an  $x$ -satisfying subgroup of  ${}_3X'$  with torsionfree  ${}_3X'/(M \cap V)$  and  $M_x \subseteq \{M_x^x\} \subseteq M \cap V$ . Hence  $M \cap V$  belongs to  $\Sigma_x$ ; and we deduce from the maximality of the rank of  $M_x$  that  $M_x$  and  $M \cap V$  have the same finite rank. Since  $(M \cap V)/M_x$  is torsionfree, it follows that  $M_x = M \cap V$ ; and this implies that  $M_x = M_x^x$  is a normal subgroup of  $X$ . Hence  $M_x$  belongs to  $\Sigma$ ; and from  $M_x \subseteq M$  and the minimality of the rank of  $M$  we conclude that  $M$  and  $M_x$  are torsionfree abelian groups of the same finite rank. Since  $M/M_x \subseteq {}_3X'/M_x$  is torsionfree, it follows that  $M = M_x$ . Thus we have shown that

$M$  is  $x$ -satisfying for every  $x$  in  $X$ .

But then every subgroup of  $M$  is normalized by every  $x$  in  $X$  so that every subgroup of  $M$  is a normal subgroup of  $X$ . Since  $M \neq 1$ , there exists consequently a cyclic normal subgroup  $Z$  of  $X$  with  $1 \subset Z \subseteq M \subseteq {}_3X'$ , completing the proof of (3).

Now it is easy to complete the proof of our theorem. Consider an epimorphic image  $H \neq 1$  of  $G$ . If  $H' = 1$ , then  $H$  is abelian and every cyclic subgroup, not 1, of  $H$  is a normal subgroup of  $H$ . Hence we assume next that  $H' \neq 1$ . It is a consequence of (1) that the epimorphic image  $H'$  of  $G'$  is hypercentral and of finite abelian rank. Hence  ${}_3H' \neq 1$  and all the components of the abelian subgroup  ${}_3H'$  of  $H$  are of finite rank.

*Case 1:*  ${}_3H'$  is torsionfree.

An immediate application of (3) shows the existence of a cyclic normal subgroup  $Z$  of  $H$  with  $1 \subset Z \subseteq {}_3H'$ .

*Case 2:*  ${}_3H'$  is not torsionfree.

Then  ${}_3H'$  contains elements of order a prime  $p$ . Denote by  $P$  the totality of elements  $x$  in  ${}_3H'$  with  $x^p = 1$ . Then  $P$  is a characteristic elementary abelian  $p$ -subgroup, not 1, of  ${}_3H'$ . Since the rank of the  $p$ -component of  ${}_3H'$  is finite, so is the rank of  $P$ . Hence  $P$  is finite. Apply (2) to show the existence of a cyclic normal subgroup  $Z$  of  $H$  with  $1 \subset Z \subseteq P$ .

Thus we have shown in all cases that  $H$  possesses a cyclic normal subgroup, not 1, proving the supersolubility of  $G$ .

**Corollary 1:** *The group  $G$  is noetherian and supersoluble if, and only if,  $G$  is locally supersoluble and its abelian subgroups are finitely generated.*

**Proof:** If  $G$  is locally supersoluble and its abelian subgroups are finitely generated, then we deduce from F, Theorem that  $G$  is supersoluble. We may apply again a generalization of a theorem of Mal'cev — see BAER [4; p. 173, Hauptsatz 4] — to show that  $G$  is noetherian.

**Corollary 2:** *Every characteristic supersoluble subgroup of a group of finite abelian subgroup rank is contained in a maximal characteristic supersoluble subgroup.*

**Proof:** If  $C$  is a characteristic supersoluble subgroup of the group  $G$  of finite abelian subgroup rank, then  $C$  is contained in a maximal

characteristic locally supersoluble subgroup  $M$  of  $G$  [Maximum Principle of Set Theory]. But  $M$  is as a subgroup of  $G$  of finite abelian subgroup rank. Application of the preceding Theorem shows the supersolubility of  $M$ .

**Remark:** In the same way one may prove that every supersoluble normal subgroup of a group of finite abelian subgroup rank is contained in a maximal supersoluble normal subgroup. But most comprehensive supersoluble normal subgroups need not exist. There exist examples of supersoluble normal subgroups of finite groups whose product is not supersoluble; see BAER [8; p. 186, Example 1]. See in this context section J below.

**Example:** Denote by  $p$  some fixed prime. By Dirichlet's Prime Number Theorem there exists an infinite set  $\mathfrak{p}$  of primes  $q$  with  $q \equiv 1$  modulo  $p$ . Denote by  $Z(q)$  a cyclic group of order  $q$  and denote by

$$A = \prod_{q \in \mathfrak{p}} Z(q)$$

the [ordinary or restricted] direct product of these groups  $Z(q)$ .

Since  $p$  is a divisor of  $q-1$ , there exists an automorphism  $\sigma(q)$  of  $A$  which induces an automorphism of order  $p$  in  $Z(q)$  and the 1-automorphism in all the other factors  $Z(x)$  with  $x \neq q$ . It is clear that every  $\sigma(q)$  is an automorphism of order  $p$ , that  $\sigma(q')$  and  $\sigma(q'')$  commute for  $q', q''$  in  $\mathfrak{p}$ , and that consequently the group  $\Sigma$  of automorphisms of  $A$ , generated by the  $\sigma(q)$  for  $q$  in  $\mathfrak{p}$ , is an elementary abelian  $p$ -group. There exist elements  $z(q)$  with  $Z(q) = \{z(q)\}$ . There exists a group  $G$ , obtained by adjoining elements  $s(q)$  for  $q$  in  $\mathfrak{p}$  to  $A$ , subject to the following relations:

$$\begin{aligned} s(q)^p &= 1, \\ s(q)^{-1} x s(q) &= x^{\sigma(q)} \text{ for } x \text{ in } A \text{ and } q \text{ in } \mathfrak{p}, \\ s(q') \circ s(q'') &= z(q') z(q'')^{-1} \text{ for } q', q'' \text{ in } \mathfrak{p}. \end{aligned}$$

This follows, for instance, from ZASSENHAUS [p. 96/97].

Clearly  $G$  is an extension of  $A$  by  $\Sigma$ ; and since  $\Sigma$  is an infinite elementary abelian  $p$ -group, the group  $G$  is not of finite abelian rank. Since every factor  $Z(q)$  of  $A$  is a cyclic characteristic subgroup of the normal subgroup  $A$  of  $G$ , and since  $G/A$  is abelian, it follows that  $G$  is a supersoluble torsiongroup. Finally one verifies that every abelian  $p$ -subgroup of  $G$  is cyclic of order  $p$  and that every primary subgroup of  $A$  is likewise cyclic. Hence  $G$  is of finite abelian subgroup rank.

This example shows the impossibility of proving an analogue to B, Lemma for supersoluble groups.

(To be continued)